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Bessel equation solution

Bessel’s equation of order \(u\) is given by \[x^2 y'' + x y' + (x^2-u^2) y = 0.\] Clearly \(x=0\) is a regular singular point, so we can solve by Frobenius’ method. The indicial equation is obtained from the lowest power after the substitution \(y=x^\gamma\), and is \[\gamma^2-u^2=0\] So a generalized series solution gives two independent solutions if \(u eq \frac{1}{2} n\). Now let us solve the problem and explicitly substitute the power series, \[y = x^u \sum_n a_n x^n.\] From Bessel’s equation we find \[\sum_n(n+u)(n+u-1) a_u x^{m+u} +\sum_n(n+u)a_u x^{m+u} +\sum_n(x^2-u^2)a_u = 0\] which leads to \[[(m+u)^2-u^2] a_m= -a_{m-2}\] or \[a_m= -\frac{1}{m(m+2u)}a_{m-2}.\] If we take \(u=n>0\), we have \[a_m= -\frac{1}{m(m+2n)}a_{m-2}.\] This can be solved by iteration, \[\begin{aligned} a_{2k} &= -\frac{1}{4}\frac{1}{k(k+n)}a_{2(k-1)}\onumber\\ &= \left(\frac{1}{4}\right)^2\frac{1}{k(k-1)(k+n)(k+n-1)}a_{2(k-2)} \onumber\\ &= \left(-\frac{1}{4}\right)^k\frac{n!}{k!(k+n)!}a_{0}.\end{aligned}\] If we choose1 \(a_0 = \frac{1}{n!2^n}\) we find the Bessel function of order \(n\) \[J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}.\] There is another second independent solution (which should have a logarithm in it) with goes to infinity at \(x=0\). Figure \(\PageIndex{1}\): A plot of the first three Bessel functions \(J_n\) and \(Y_n\). The general solution of Bessel’s equation of order \(n\) is a linear combination of \(J\) and \(Y\), \[y(x) = A J_n(x)+B Y_n(x).\] Consider the following differential equation (called a Bessel equation) $$t\frac{d^2 y}{dt^2}+\frac{dy}{dt}+ty=0$$ It is impossible to solve this equation in terms of simple functions, but it is possible to find a solution as a power series. Why not try to solve it to get a feel for the problems involved? Could you explain clearly to someone why the equation is hard to solve? Although we can't solve it in terms of simple functions, we could look at the equation for very large or very small values of t to try to get an approximate solution at those times. Think about these suggestions: 1. EITHER $y(t)$ or its derivatives are very large for large time t OR y approximately satisfies a simple harmonic motion equation for very large time t. 2. EITHER y or its derivatives are very large for very small time t OR y is approximately constant for very small time $n!$. Can you see why these possibilities might make sense? A power series analysis shows that one of the solutions to the equation is $$y_0(t) = 1-\frac{1}{(1!)^2}\left(\frac{t}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{t}{2}\right)^4-\frac{1}{(3!)^2}\left(\frac{t}{2}\right)^6+\frac{1}{(4!)^2}\left(\frac{t}{2}\right)^8 + \ldots$$ [Note: $n!$, called 'n-factorial', means $n\times(n-1)\times(n-2)\times \dots \times 2\times 1$] By working out $t\frac{d^2 y_0}{dt^2}$, $\frac{dy_0}{dt}$ and $ ty_0$ separately, show that the terms of like powers for the first few powers of t all cancel out exactly if we substitute this value into the Bessel equation. If this series were to continue in a regular fashion (which it does), what would the next term in the expansion be? Can you easily determine for sure the behaviour of the function at very small times from the power series? Can you easily determine for sure the behaviour of the function at very large times from the power series? The first 150 terms of the series were plotted on a spreadsheet, resulting in the following two charts: the first is plotted on the range $0\leq x \leq 40$ and the second on the range $0\leq x \leq 45$ What has gone wrong? NOTES AND BACKGROUND Bessel was a 19th century astronomer and the first person accurately to predict the distance to a star: using the method of parallax he calculated the 61 Cygni was a mere 9.8 light years away. In engineering and physics, Bessel's equation is of similar importance to the equation for simple harmonic motion; the solutions to Bessel's equation, called Bessel functions, may be though of as time-decaying versions of sine and cosine. They are both theoretically interesting and practically useful. This section concerns about Bessel equation and its solutions, known as Bessel functions. Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessels functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain suspended at one end. In 1764 Leonhard Euler employed Bessel functions of both zero and integral orders in an analysis of vibrations of a stretched membrane, an investigation which was further developed by Lord Rayleigh in 1878, where he demonstrated that Bessels functions are particular cases of Laplaces functions. Bessel, while receiving named credit for these functions, did not incorporate them into his work as an astronomer until 1817. The Bessel function was the result of Bessels study of a problem of Kepler for determining the motion of three bodies moving under mutual gravita- tion. In 1824, he incorporated Bessel functions in a study of planetary perturbations where the Bessel functions appear as coefficients in a series expansion of the indirect perturbation of a planet, that is the motion of the Sun caused by the perturbing body. It was likely Lagrange’s work on elliptical orbits that first suggested to Bessel to work on the Bessel functions. Return to computing page for the first course APMA0330 Return to computing page for the second course APMA0340 Return to Mathematica tutorial for the first course APMA0330 Return to Mathematica tutorial for the second course APMA0340 Return to the main page for the first course APMA0330 Return to the main page for the second course APMA0340 Return to Part VII of the course APMA0340 Introduction to Linear Algebra with Mathematica The Bessel differential equation is the linear second-order ordinary differential equation given by \[x^2 \frac{{\text d}^2 y}{{\text d} x^2} + x\,\frac{{\text d} y}{{\text d} x} + \left(x^2 - u^2 \right) y(x) = 0 \qquad \mbox{or} \qquad \frac{\text d}{{\text d}x} \left(x \,\frac{{\text d} y}{{\text d} x} \right) + \left(x - \frac{u^2}{x} \right) y(x) = 0 , \] where ν is a real constant, called the order of the Bessel equation. Upon substitution \(y = u\, x^{-1/2} , \) it is reduced to the equation without first derivative: \[u'' + \left(1 + \frac{1 - 4u^2}{4x^2} \right) u(x) = 0 . \] A Bessel equation is a special case of a confluent hypergeometric equation. Since $x = 0$ is a regular singular point for the Bessel equation, one of its solution can be bounded at this point but another linearly independent solution should be unbounded. Transformed version of the Bessel equation The self-adjoint differential equation \[\frac{\text d}{{\text d}x} \left(x^a \,\frac{{\text d} y}{{\text d} x} \right) + b\, x^c y(x) = 0 , \] where a, b, and c are real numbers, can be transformed to a Bessel equation by transforming both independent and dependent variables. Because of the powers of x in the differential equation, it seems promissing to change variables from x, y(x) to t, u(t) according to \[t = A\, x^B , \qquad u = x^C y , \] and to try to find A, B, C so that the new equation becomes a Bessel equation for u(t). It turns out that the plan works and one finds that under the change of variables \[t = \alpha \sqrt{b} \,x^{1/\alpha} \qquad \mbox{and} \qquad u = x^{-u /\alpha} y \] the given self-adjoint equation is transformed into the Bessel equation of order ν, \[t^2 \frac{{\text d}^2 u}{{\text d} t^2} + t\,\frac{{\text d} u}{{\text d} t} + \left(t^2 - u^2 \right) u(t) = 0 , \] if we choose \[\alpha = \frac{2}{c-a+2} \qquad\mbox{and} \qquad u = \frac{1-a}{c-a+2} . \] The latter is meaningless when c - a + 2 = 0, but in that case the original equation \[\frac{\text d}{{\text d}x} \left(x^a \,\frac{{\text d} y}{{\text d} x} \right) + b\, x^{a-2} y(x) = 0 \] becomes an Euler equation \[x^{a-2} \left[x^2 \frac{{\text d}^2 y}{{\text d}x^2} + ax \, \frac{{\text d}y}{{\text d}x} + b\,y(x) \right] = 0 . \] Thus, if \[Z_{u} (x) = C_1 J_{u} (x) + C_2 Y_{u} (x) \] denotes the general solution of the Bessel equation with arbitrary constants C1 and C2, then the general solution of the given self-adjoint equation becomes \[y(x) = x^{u /\alpha} Z_{u} \left(\alpha \sqrt{|b|} \, x^{1/\alpha} \right) . \] If $b < 0$, then we have to choose \[Z_{u} (x) = C_1 I_{u} (x) + C_2 K_{u} (x) \] as the general solution to the modified Bessel equation. In the case of the general Bessel equation given by Bowman (1958) \[x^2 \frac{{\text d}^2 y}{{\text d}x^2} + \left(2p +1 \right) \frac{{\text d}y}{{\text d}x} + \left(\alpha^2 x^{2r} + \beta^2 \right) y =0 , \] its solution is \[y(x) = x^{-p} \left[C_1 J_{q/r} \left(\frac{\alpha}{r}\,x^r \right) + C_2 Y_{q/r} \left(\frac{\alpha}{r}\,x^r \right) \right] , \] where \[q = \sqrt{p^2 - \beta^2} , \] Jn(x) and Yn(x) are the Bessel functions of the first and second kind, and C1 and C2 are arbitrary constants. Another form is given by letting \(y= x^{\alpha} J_n \left(\beta\, x^{\gamma} \right) , \quad \eta = y\,x^{\gamma} , \ \mbox{ and } \ \xi = \beta\,x^{\gamma} \) (see Bowman, 1958, p. 117), then \[\frac{{\text d}^2 y}{{\text d}x^2} - \frac{2\alpha -1}{x} \, \frac{{\text d}y}{{\text d}x} + \left(\beta^2 \gamma^2 x^{2\gamma -2} + \frac{\alpha^2 - n^2 \gamma^2}{x^2} \right) y(x) = 0 . \] The solution is \[y(x) = \begin{cases} x^{\alpha} \left[C_1 J_n \left(\beta\, x^{\gamma} \right) + C_2 Y_n \left(\beta\, x^{\gamma} \right) \right] \quad \mbox{for integer } n , \\ x^{\alpha} \left[C_1 J_n \left(\beta\, x^{\gamma} \right) + C_2 J_{-n} \left(\beta\, x^{\gamma} \right) \right] \quad \mbox{for noninteger } n , \end{cases} \] Bessel function of first kind The Bessel function of first kind of order ν is \[J_{u} (x) = \sum_{k\ge 0} \frac{(-1)^k}{k!\,\Gamma (k+u +1)} \left(\frac{x}{2} \right)^{2k+u} , \] Bessel functions of the first kind $J_\nu(x)$ are bounded at the origin for any ν. When ν is not an integer, Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent because their Wronskian is \[W\left[J_{u} (x), J_{-u} (x) \right] = J_{u +1} (x)\,J_{-u} + J_{u } (x) (x)\,J_{-u -1} (x) = -\frac{2\,\sin \left(u \pi \right)}{\pi\, x} . \] When $\nu = n$ an integer, $J_{-n} (x) = (-1)^n J_n (x) .$ \] plot = Plot[{BesselJ[0, x], BesselJ[1, x], BesselJ[2, x]}, {x, 0.01, 14}, AxesLabel -> {x, BesselJ, PlotStyle -> {Thick}, PlotLegends -> "Expressions"]; left = Graphics[Text[Style["\!\(*SubscriptBox[\(J\), \(0\)]\)(x)", {1.5, 0.95}]]; center = Graphics[Text[Style["\!\(*SubscriptBox[\(J\), \(1\)]\)(x)", {2.2, 0.65}]]; right = Graphics[Text[Style["\!\(*SubscriptBox[\(J\), \(2\)]\)(x)", {4.0, 0.55}]]; Show[plot, left, center, right] Graphs of Bessel functions of first kind. \begin{aligned*} J_n (x) &= \frac{1}{\pi} \int_0^{\pi} \cos \left(n\theta -x\,\sin\theta \right) {\text d}\theta , \\ J_1 (x) &= \frac{1}{\pi} \int_0^{\pi} \cos \left(\theta - x\,\sin\theta \right) {\text d}\theta = \frac{1}{\pi} \int_0^{\pi} \cos \left(\theta \,\sin \left(x\,\cos\theta \right) \right) {\text d}\theta , \\ J_0 (x) &= \frac{1}{\pi} \int_0^{\pi} \cos \left(x\,\sin\theta \right) {\text d}\theta = \frac{1}{\pi} \int_0^{\pi} \cos \left(x\,\cos\theta \right) {\text d}\theta . \end{aligned*} \begin{aligned*} J_{u +1} (x) &= \frac{2u}{x}\, J_{u} (x) - j_{u -1} (x) , \end{aligned*} \begin{aligned*} J'_{u +1} (x) &= \frac{1}{2} \left[J_{u -1} (x) - J_{u +1} (x) \right] , \\ J'_{u} (x) &= J_{u -1} (x) - \frac{u}{x}\, J_{u} (x) , \\ J'_{u} (x) &= \frac{u}{x}\, J_{u} (x) - J_{u +1} (x) , \\ \frac{\text d}{{\text d}x}\, \left[x^{u} J_{u} (x) \right] &= x^{u} J_{u -1} (x) , \\ \frac{\text d}{{\text d}x} \left[x^{-u} J_{u} (x) \right] &= -x^{-u} J_{u +1} (x) . \end{aligned*} Bessel function of second kind The Bessel function of the second kind of order ν is defined as follows: \[Y_{u} (x) = \frac{\cos u \pi \, J_{u} (x) - J_{-u} (x)}{\sin u \pi} . \] This function is called either after Heinrich Martin Weber (1875--1913), who introduced $Y_\nu(x)$ in 1873, or after Carl Neumann (1832--1925), with notation $N_\nu(x)$. Since there is no standard notation, we will distinguish them by defining the Neumann function $N_\nu(x) = \pi Y_\nu(x)$ as π multiple of the Weber function: \[N_{u} (x) = \pi\,Y_{u} (x) . \] These functions are clearly solutions of the Bessel equation, as they are linear combinations of Bessel functions of the first kind. However, if ν is an integer, then $Y_\nu(x)$, as defined, is the indeterminate form 0/0. Therefore, we need to use l’Hospital’s Rule to determine whether the limit as ν approaches an integer n is nonzero, so that we can obtain a meaningful solution. Applying l’Hospital’s Rule yields \[\begin{split} Y_{u} (x) &= \lim_{u \to n} \,\frac{\cos u \pi \, J_{u} (x) - J_{-u} (x)}{\sin u \pi} \\ &= \lim_{u \to n} \,\frac{-\pi\,\sin u\pi \,J_{u} (x) + \cos u \pi \, \frac{\text d}{{\text d}u} \, J_{u} (x) - \frac{\text d}{{\text d}u} \, J_{-u} (x)}{\pi\,\cos u\pi} \\ &= \lim_{u \to n} \,\frac{1}{\pi} \left[\frac{\text d}{{\text d}u} \, J_{u} (x) - \frac{\text d}{{\text d}u} \, J_{-u} (x) \right] . \end{split} \] Using the series definition of the Bessel function of the first kind, \[J_{u} (x) = \sum_{k\ge 0} \frac{(-1)^k}{k!\,\Gamma (k+ u +1)} \left(\frac{x}{2} \right)^{2k+u} , \] as well as the digamma function \[\psi (x) = \frac{\text d}{{\text d}x}\, \ln \Gamma (x) = \frac{\Gamma' (x)}{\Gamma (x)} , \] we obtain \[\begin{split} Y_{n} (x) &= \frac{1}{\pi} \left[\sum_{k\ge 0} \frac{(-1)^k}{k! \, (k+n)!} \left(\frac{x}{2} \right)^{2k+n} \ln \left(\frac{x}{2} \right) + (-1)^n \sum_{k\ge 0}\frac{(-1)^k}{k! \, (k-n)!} \left(\frac{x}{2} \right)^{2k-n} \ln \left(\frac{x}{2} \right) \right] \\ &\quad - \frac{1}{\pi} \left[\sum_{k\ge 0} \frac{(-1)^k \Gamma' (k+n+1)}{k! \,(k+n)!^2} \left(\frac{x}{2} \right)^{2k+n} + (-1)^n \sum_{k\ge 0} \frac{(-1)^k \Gamma' (k-n+1)}{k! \,(k-n)!^2} \left(\frac{x}{2} \right)^{2k-n} \right] \\ &= \frac{2}{\pi} \left[J_n (x), \ln \left(\frac{x}{2} \right) + (-1)^n J_{-n} (x), \ln \left(\frac{x}{2} \right) \right] \\ &\quad - \frac{1}{\pi} \left[\sum_{k\ge 0} \frac{(-1)^k \psi (k+n+1)}{k!\,(k+n)!} \left(\frac{x}{2} \right)^{2k+n} + (-1)^n \sum_{k\ge -n} \frac{(-1)^{k+n} \psi (k+1)}{k!\,(k+n)!} \left(\frac{x}{2} \right)^{2k+n} \right] \\ &= \frac{2}{\pi} \, J_n (x)\, \ln \left(\frac{x}{2} \right) - \frac{(-1)^n}{\pi} \,\sum_{k=-n}^{-1} \frac{(-1)^k \psi (k+1)}{k!\,(k+n)!} \left(\frac{x}{2} \right)^{2k+n} \\ &\quad - \frac{1}{\pi} \sum_{k\ge 0} \frac{(-1)^k}{k! \,(k+n)!} \left(\frac{x}{2} \right)^{2k+n} \left[\psi (k+n+1) + \psi (k+1) \right] \\ &= \frac{2}{\pi} \, J_n (x)\, \ln \left(\frac{x}{2} \right) - \frac{1}{\pi} \,\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} \\ &\quad - \frac{1}{\pi} \sum_{k\ge 0} \frac{(-1)^k}{k!\, (k+n)!} \left(\frac{x}{2} \right)^{2k+n} \left[\psi (k+n+1) + \psi (k+1) \right] , \end{split} \] where we have used the limit \[\lim_{z\to -n} \,\frac{\psi (z)}{\Gamma (z)} = (-1)^{n-1} n! \, . \] We can see from the above expression for Yn(x) that it is indeed linearly independent of Jn(x), so that we have two linearly independent solutions of the Bessel equation for integer order n. Also, unlike Jn(x), the Weber function is singular at x = 0. Using the identity \(\psi (n) = H_{n-1} - \gamma , \) we have \begin{aligned*} Y_0 (x) &= \frac{2}{\pi} \left[\gamma + \ln \left(\frac{x}{2} \right) \right] J_0 (x) - \frac{2}{\pi} \sum_{k\ge 0} \frac{(-1)^k}{k!\, k!} \left(\frac{x}{2} \right)^{2k} H_k , \\ Y_n (x) &= \frac{2}{\pi} \left[\gamma + \ln \left(\frac{x}{2} \right) \right] J_n (x) - \frac{1}{\pi} \sum_{k= 0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} - \frac{1}{\pi} \sum_{k\ge 0} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2} \right)^{2k} \left[H_k + H_{k+n} \right] , \end{aligned*} where $\gamma \approx 0.5772156649...$ is the Euler constant (also called the Euler-Mascheroni constant in honor of Lorenzo Mascheroni (1750--1800)), which is implemented in the Wolfram Language as EulerGamma, and Hn is the n-th harmonic number: \[H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^n \frac{1}{k} . \] All Weber functions are unbounded at the origin, as it is seen from the plot. Plot[{BesselY[0, x], BesselY[1, x], BesselY[2, x]}, {x, 0.01, 14}, AxesLabel -> {x, BesselJ, PlotStyle -> {Thick}, PlotLegends -> "Expressions"] Graphs of Weber functions. The Wronskian is \[W \left[J_{u} (x), Y_{u} (x) \right] = J_{u +1} (x)\,Y_{u} (x) - J_{u} (x)\,Y_{u +1} (x) = \frac{2}{\pi\,x} . \] Bessel functions of the second kind also have integral representations. We have \begin{aligned*} Y_0 (x) &= - \frac{2}{\pi} \int_0^{\infty} \cos \left(x\,\cosh t \right) {\text d} t , \\ Y_n (x) &= \frac{1}{\pi} \int_0^{\pi} \sin \left(x\,\sin t - nt \right) {\text d} t - \frac{1}{\pi} \int_0^{\infty} \left[e^{nt} +(-1)^n e^{-nt} \right] e^{-x\,\sinh t} {\text d} t . \end{aligned*} Bessel functions of the second kind, being solutions of the Bessel equation, satisfy the same recurrence relations as the Bessel functions of the first kind. Specifically, \begin{aligned*} Y_{u -1} (x) - Y_{u +1} (x) &= 2\,Y_{u} (x) , \\ Y_{u -1} (x) + Y_{u +1} (x) &= \frac{2u}{x}\, Y_{u} (x) . \end{aligned*} We also have the relation \[Y_{-n} (x) = (-1)^n Y_n (x) , \] when n is an integer. Bessel function of third kind We can define two new linearly dependent functions, called the Hankel functions \begin{aligned*} H_{u}^{(1)} (x) &= J_{u} (x) + {\bf j}\,Y_{u} (x) = \frac{J_{-u} (x) - e^{-u\pi{\bf j}} J_{u} (x)} }{-{\bf j}\,\sin u\pi} , \\ H_{u}^{(2)} (x) &= J_{u} (x) - {\bf j}\,Y_{u} (x) = \frac{J_{-u} (x) - e^{u\pi{\bf j}} J_{u} (x)} }{-{\bf j}\,\sin u\pi} , \end{aligned*} where j is a unit vector in positive vertical direction on the complex plane $\mathbb{C}$ so that $j^2 = -1$. Grigoryan, V, Parial Differential Equations, 2010, University of California, Santa Barbara. Olver, E.W.J., Maximon, L.C., Chapter 10 Bessel Functions, National Institute of Standards and Tech-nology, 2010. 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